

On the nature of finite groups

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13.08.2004

Abstract

The reality of the difficulties in an investigation of finite groups are considered. It is shown that the consideration of symmetry properties of the k -orbits that are obtained with an action of a finite group $F = (V, \cdot)$ on Cartesian power V^k gives a new view on the nature of groups and simplifies some difficult properties of groups.

Using this representation it is obtained a simple proof of the W. Feit, J.G. Thompson theorem: Solvability of groups of odd order.

Key words: k -orbits, partitions, permutations, symmetry, groups

1 Introduction

The group theory was born as the permutation group theory and later was abstracted to a group algebra with corresponding properties of a group operation.

Let F be an abstract group on a set F , then the group algebra is equivalent to the action of F on F . But this algebra generates also the action of F on F^k . It is clear that that action can have its own properties which belong of course also to group properties, but those properties lie out of group algebra.

Namely a consideration of such properties joined with a term *k-orbit theory* was first considered by author in [1]. A full text of *k-orbit theory* with applications is planned to be published later. Here we consider how this theory leads to a simple proof of the W. Feit, J.G. Thompson theorem: Solvability of groups of odd order (it is known that the original text covers 255 pages [3]).

Below under primitive permutation group we understand non-Abelian primitive group.

2 n -Orbit representation of finite groups

Let F be a finite group, $A < F$, $|F|/|A| = n$ and \vec{L}_n, \vec{R}_n be ordered sets of left, right cosets of A in F . It is known that every transitive permutation representation of F is equivalent to a representation of F given by *n-orbits* $X'_n = \{f\vec{L}_n : f \in F\}$ or $X''_n = \{\vec{R}_n f : f \in F\}$. It is also known that F is homomorphic to its image $\text{Aut}(X'_n) = \text{Aut}(X''_n)$ with the kernel of the homomorphism equal to a maximal normal subgroup of F contained in A . Further we assume that a finite group is isomorphic to its permutation (= *n-orbit*) representation.

A maximal by inclusion subgroup A of a finite group F that contains no normal subgroup of F we call a *md-stabilizer* of F and a corresponding representation of F we call a *md-representation* or a *md-group*. A md-stabilizer A of a finite group F defines a minimal degree permutation representation of F in a family of permutation representations of F produced with subgroups of A . A non-minimal degree representation of F we call a *nmd-representation* or a *nmd-group*. A finite group can have many (non-conjugate) md-stabilizers.

A special interest is presented by a permutation representation of the lowest degree or a *ld-representation* or a *ld-group*, because a n -orbit of such representation (of degree n) and its k -projections (k -orbits) contain all specific symmetry properties which describe a finite group. This representation not always can be represented with permutations of cosets of some subgroup of finite group. As distinct from *ld-representation*, k -orbits of other permutation representations can have additional symmetry properties that are not specific for a finite group, but describe properties of corresponding permutation group as for example the automorphism group of Petersen graph. Some properties of *ld-representation* (and also other representations) are on principle combinatorial and cannot be interpreted with group algebra. For example, a *ld-representation* of a direct product and only that representation is intransitive. One more example: the n -orbit is combinatorially a $|F| \times n$ matrix in which are arranged n elements of a base set V . This combinatorics leads to invariants of the group that are not existing in group algebra. The study of these invariants gives a hope to obtain a simple full invariant for a big class of groups. For example it is of interest the next

Hypothesis 1 *A primitive ld-group is defined by its order and degree.*

or

Problem 2 *Are there existing two primitive md-groups of a degree n with the same order but non-isomorphic n -orbits?*

For imprimitive groups there exist many such examples. We can also note that a *nmd-group* is always imprimitive.

Now we consider one property that is related with a structure of permutation groups. In [2] is described a hypothesis (polycirculant conjecture) of M. Klin and D. Marušič that the automorphism group of a transitive graph contains a regular element (a permutation decomposed in cycles of the same length). On group theory language this conjecture has the next interpretation.

Let F be a finite group and $A < F$ be a *md-stabilizer*. Let F contains a subgroup P of a prime order p that is intersected with no subgroup from stabilizers of F conjugated with A , then it follows that P is a regular subgroup of a representation $F(F/A)$. So the polycirculant conjecture statements that if $F(F/A)$ is a graph automorphism group, then F contains the corresponding subgroup P .

The attempts to use such approach was not successful. The reason is that the such interpretation of the problem is out of the inside structure of a n -orbit, but the desired property is directly connected with symmetries of n -orbits of corresponding permutation groups (s.[1]).

It is known the existence of external and internal automorphisms of finite groups. The similar property also exists in n -orbits. If X_n is a n -orbit of a permutation group G of a degree n , then it can contain isomorphic k -subsets that are connected with permutation of G or with permutation of symmetric group S_n (the simplest example of the latter is given by intransitive groups). This property of a n -orbit has only partial interpretation in group theory, but it plays an important role in construction of a n -orbit and hence in construction of the related group.

The next is a combinatorial property of some normal subgroups.

Lemma 3 *Let $G(V)$ be an imprimitive md-group and Q be a partition of V on imprimitivity blocks (i.e. the action of G on V maintains Q), then the maximal subgroup $Stab(\vec{Q}) < G$ that maintains all classes of Q (i.e. arbitrarily ordered set Q) is a normal subgroup of G .*

Proof: Subgroups $\{Stab(U) < G : U \in Q\}$ form a class of conjugate subgroups and $Stab(\vec{Q}) = \bigcap_{(U \in Q)} Stab(U)$. For a *md-group* $Stab(\vec{Q})$ is not trivial, because on definition an imprimitive *md-group* has not isomorphic representation on Q . \square

From here immediately follows that a *md-representation* of a simple group is primitive. A simple proof of the Feit-Thompson theorem follows then from the following statement.

Theorem 4 *Let $G(V)$ be a primitive group of odd order, then it contains a normal subgroup.*

Now we shall prove this theorem.

Lemma 5 *Any permutation from G fixes at most one element of V .*

Proof: First, we can assume that no subgroup of G that has a primitive representation contradicts with lemma. Second, if a permutation g of G fixes k element of V (i.e. its decomposition in cycles contains exactly k cycles of the length 1), then k is an *automorphic number*, i.e. G contains a k -element *suborbit* (or some subgroup of G contains a k -element orbit). Moreover, if g fixes k -tuple α_k , then a set $Co(\alpha_k)$ of coordinates of α_k is a suborbit of G .

Let for G the lemma be not correct and $k > 1$ be a number of elements of V which are fixed by permutation g of G . Let I_k be fixed by g k -subspace (k -tuple), X_n be a n -orbit of G and X_k be a projection of X_n on I_k ($X_k = \hat{p}(I_k)X_n = GI_k$), then $|X_k| > n$, so there exist permutations of G that fix only one element of V .

Let a stabilizer $Stab(v_1)$ of some element $v_1 \in V$ has non-regular transitive representation $A(U^1)$ on subset U^1 of V (i.e. $|A(U^1)| > |U^1|$), then a stabilizer of $A(U^1)$ fixes certain odd number of elements from U^1 . So in order to fix odd number of elements of V with permutations of $Stab(v_1)$ there must exist a subdirect product $A(U^1) \circ \dots \circ A(U^l)$ on an even number of non-intersected subsets U^1, \dots, U^l , where $\{v_1\} \cup U^1 \cup \dots \cup U^l = V$. Then there exists only two possibilities. First, subsets U^1, \dots, U^l are G -isomorphic, then G contains a subgroup whose projection on $V \setminus \{v_1\}$ is imprimitive and hence G contains an involution. Contradiction. In second we consider a fixed k -tuple α_k that contains all fixed elements from U^1, \dots, U^l and v_1 . A stabilizer $B = Stab(\alpha_k)$ has non-trivial normalizer $N = Stab(Co(\alpha_k))$. From here follows that projection of B on $V \setminus \{v_1\}$ has non-trivial normalizer that is a projection of N on $V \setminus \{v_1\}$. It is only possible if for $T^1 = Co(\alpha_k) \cap U^1$ a subset $Z^1 = U^1 \setminus T^1$ is automorphic. Since $|Z^1|$ is even then we have the contradiction again. \square

Corollary 6 *A stabilizer $A < G$ of an element $v \in V$ has regular representation on a subset of V and hence $|A| < n$.*

Corollary 7 *G contains a regular subgroup H .*

Proof: Stabilizers of G have trivial intersections, hence G contains $n - 1$ element which belong to no stabilizer. All this elements are regular, because if one of those is not regular, then it generates a permutation that fixes more than one element of V . \square

Corollary 8 *G is ld -group.*

Proof: It follows from obtained above the structure of G . \square

So we have that $|H| > |A|$ and hence a representation of G on classes of G/H is a homomorphism. It follows that H is a normal subgroup of G .

Corollary 9 *Let F be a simple group, then its order is divisible by 4.*

Proof: If $|F| = 2m$ and m is odd, then F has evidently a normal subgroup. \square

Conclusion

A specificity of the n -orbit representation is a possibility to do a group visible. In order to go to this visibility one makes a partition of a matrix of a n -orbit on cells of k -orbits of subgroups of an investigated permutation group and studies symmetry properties of cells and the whole partition. It is not difficult to find that there exists only three kinds of basic cells which allow to construct a n -orbit of any group. Namely this approach to an investigation of permutation groups gives the progress in some cases that are difficult for other existing methods.

The using of k -orbits symmetry properties to the polynomial solution of the graph isomorphism problem was begun by Author in 1984. The generalization of k -orbits (regular k -sets) was used for describing of the structure of strongly regular graphs and their generalization on dimensions greater than two.

In 1997 Author understood the connection between the graph isomorphism problem and the problem of a full invariant of a finite group and did some attempts to obtain this full invariant by constructing of some appropriate group representations. But again the best approach was obtained with k -orbit representation.

Then this method was applied to the polycirculant conjecture and again with success.

Thus all these themes wait for a possibility to be published.

Author will be thankful for any contact related with mentioned themes and problems that are difficult in finite group theory and permutation group theory.

Acknowledgements

I would like to express many thanks to Prof. P. Cameron for his excellent site on the internet, for two years of e-mail contacts, for presenting me some examples of permutation groups interesting for analysis by methods of k -orbit theory, and for pointing out mistakes in my first version of this text.

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